THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5540 Advanced Geometry 2016-2017
Supplementary Exercise 5

1. Let $z_1 = \dfrac{2\sqrt{2}-\sqrt{3}}{2} + \dfrac{i}{2}$ and $z_2 = \dfrac{2\sqrt{2}-\sqrt{3}}{2} - \dfrac{i}{2}$ be two points on $\mathbb{D}$.

   (a) Find the equation of P-line passing through z_1 and z_2 and express your answer in form of $(x-h)^2 + (y-k)^2 = r^2$.

   (b) Find the distance $d(z_1, z_2)$ between z_1 and z_2 with respect to the Poincaré metric.

2. Let $A = z_1 = \dfrac{1}{3}$, $B = z_2 = 0$ and $C = z_3 = \dfrac{i}{2}$ be three points on $\mathbb{D}$. Find the P-angle $\angle BAC$.

3. Recall that $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

   Show that

   (a) $\cosh^2 x - \sinh^2 x = 1$

   (b) $\sinh 2x = 2\sinh x \cosh x$

   (c) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$

4. Let $t = \tanh \dfrac{x}{2}$, show that

   (a) $\sinh x = \dfrac{2t}{1-t^2}$;

   (b) $\cosh x = \dfrac{1+t^2}{1-t^2}$;

   (c) $\tanh x = \dfrac{2t}{1+t^2}$.

5. Recall the cosine rule for hyperbolic triangle:

$$\sinh b \sinh c \cos A = \cosh b \cosh c - \cosh a$$

   and the sine rule for hyperbolic triangle:

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}.$$

   (a) If $A = 30°$, $B = 45°$ and $a = \ln 4$. Find the value(s) of c.

   (b) If $\triangle ABC$ is an equilateral P-triangle where length of each side is $\ln 2$, then find an interior P-angle.

Lecturer's comment:

1. (a) The P-line passing through z_1 and z_2 is the intersection of the circle passing through z_1, z_2 and $\dfrac{1}{\overline{z_1}}$ (as well as $\dfrac{1}{\overline{z_2}}$) and $\mathbb{D}$. The required equation is

$$(x-\sqrt{2})^2 + y^2 = 1.$$

(b) Let $f(z)=\dfrac{z-z_1}{\overline{z_1}z-1}$. Then $f(z_1)=0$ and $f(z_2)=\dfrac{z_2-z_1}{\overline{z_1}z_2-1}$. We have,

$$\begin{aligned} d(z_1,z_2) &= d(0,f(z_2)) \\ &= \ln\frac{1+|f(z_2)|}{1-|f(z_2)|} \\ &= \ln\frac{1+\sqrt{11-4\sqrt{6}}}{1-\sqrt{11-4\sqrt{6}}} \end{aligned}$$

2. Let $f(z)=\dfrac{z-\frac{1}{3}}{\frac{1}{3}z-1}=\dfrac{3z-1}{z-3}$. Let A', B' and C' be the images of A, B and C under $f(z)$ respectively.

Then, $A'=f(z_1)=0$, $B'=f(z_2)=\dfrac{1}{3}$,

$C'=f(z_3)=\dfrac{15}{37}-\dfrac{16}{37}i=R(\cos\alpha+i\sin\alpha)$, where $R=\dfrac{\sqrt{481}}{37}$ and $\alpha\approx-46.8°$.

Therefore, the P-angle $\angle BAC=\angle B'A'C'\approx 46.8°$.

3. (a) $\cosh^2 x-\sinh^2 x=\left(\dfrac{e^x+e^{-x}}{2}\right)^2-\left(\dfrac{e^x-e^{-x}}{2}\right)^2=\dfrac{e^{2x}+2+e^{-2x}}{4}-\dfrac{e^{2x}-2+e^{-2x}}{4}=1$

(b) $2\sinh x\cosh x=2\left(\dfrac{e^x+e^{-x}}{2}\right)\left(\dfrac{e^x-e^{-x}}{2}\right)=2\left(\dfrac{e^{2x}-e^{-2x}}{4}\right)=\dfrac{e^{2x}-e^{-2x}}{2}=\sinh 2x$

(c) $\cosh^2 x+\sinh^2 x=\left(\dfrac{e^x+e^{-x}}{2}\right)^2+\left(\dfrac{e^x-e^{-x}}{2}\right)^2=\dfrac{e^{2x}+2+e^{-2x}}{4}+\dfrac{e^{2x}-2+e^{-2x}}{4}=\dfrac{e^{2x}+e^{-2x}}{2}=\cosh 2x$

Also, by (a), we have $\cosh 2x=\cosh^2 x+\sinh^2 x=\cosh^2 x+(\cosh^2 x-1)=2\cosh^2 x-1$ and $\cosh 2x=\cosh^2 x+\sinh^2 x=(\sinh^2 x+1)+\sinh^2 x=2\sinh^2 x+1$.

4. (a) $\dfrac{2t}{1-t^2}=\dfrac{2\left(\frac{\sinh\frac{x}{2}}{\cosh\frac{x}{2}}\right)}{1-\left(\frac{\sinh\frac{x}{2}}{\cosh\frac{x}{2}}\right)^2}=\dfrac{2\sinh\frac{x}{2}\cosh\frac{x}{2}}{\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}}=\dfrac{\sinh x}{1}=\sinh x$

(b) $\dfrac{1+t^2}{1-t^2}=\dfrac{1+\left(\frac{\sinh\frac{x}{2}}{\cosh\frac{x}{2}}\right)^2}{1-\left(\frac{\sinh\frac{x}{2}}{\cosh\frac{x}{2}}\right)^2}=\dfrac{\cosh^2\frac{x}{2}+\sinh^2\frac{x}{2}}{\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}}=\dfrac{\cosh x}{1}=\cosh x$

(c) $\tanh x=\dfrac{\sinh x}{\cosh x}=\dfrac{\left(\frac{2t}{1-t^2}\right)}{\left(\frac{1+t^2}{1-t^2}\right)}=\dfrac{2t}{1+t^2}$

5. (a) By using sine rule, $\dfrac{\sin A}{\sinh a}=\dfrac{\sin B}{\sinh b}$, so $\sinh b=\dfrac{15\sqrt{2}}{8}$ and $b=\sinh^{-1}(\dfrac{15\sqrt{2}}{8})\approx 1.702$.

Then, by using the cosine rule,

$$\begin{aligned} \sinh b\sinh c\cos A &= \cosh b\cosh c-\cosh a \\ \sinh b\cos A(\frac{e^c-e^{-c}}{2}) &= \cosh b(\frac{e^c+e^{-c}}{2})-\cosh a \\ (\sinh b\cos A-\cosh b)e^{2c}+(2\cosh a)e^c-(\sinh b\cos A+\cosh b) &= 0 \end{aligned}$$

which is a quadratic equation. Therefore, $c\approx 0.397$ or $c\approx 1.859$.

(b) Poincaré disk is a Hilbert plane, so proposition I.5 (Base angles of an isosceles triangle equal to each other) in Eucild's Elements also holds on Poincaré disk. Therefore, all interior P-angles of an equaliteral P-triangle are the same. Now, we have $a = b = c = \ln 2$. Then,

$$\begin{aligned}
\cos A &= \frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c} \\
&= \frac{\cosh^2(\ln 2) - \cosh(\ln 2)}{\sinh^2(\ln 2)} \\
&= \frac{(5/4)^2 - (5/4)}{(3/4)^2} \\
&= \frac{5}{9} \\
A &= \cos^{-1}(\frac{5}{9}) \\
&\approx 56.3^\circ
\end{aligned}$$